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ESTUDOS DO GEMF

N.º 01

2012

**PUBLICAÇÃO CO-FINANCIADA PELA
FUNDAÇÃO PARA A CIÊNCIA E TECNOLOGIA**

Impresso na Secção de Textos da FEUC
COIMBRA 2012

Consuming durable goods when stock markets jump: a strategic asset allocation approach

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December 1, 2011

Abstract

Agents derive their utilities from consumption over time. In this paper we consider an agent that invests in the financial market and in consumption goods. The agent has an infinite time horizon and a utility that depends on consumption at each point in time, consuming both a durable good and a perishable good. There are costs for transacting the durable good. We show that an agent who does not consider the impact of jumps in the return process of risky assets will make suboptimal decisions, not only regarding the fraction of wealth invested in the stock market, but also with respect to the timing for trading on the durable good.

1 Introduction

In times of financial crisis such as the one that we are living, the role of downward jumps in financial decisions becomes particularly relevant. A quick look at the performance of financial markets in the past two years makes this point clear, when the intensity of jumps has been particularly high.

In this paper we aim to analyze the impact of such downward jumps in the investment and consumption decisions, particularly when agents consume durable goods for which there are transaction costs.

Our focus in these downward jumps is natural¹. The main problem faced by investors is the uncertainty regarding their future income and capacity to consume. Such uncertainty is typically characterized by the first two moments of the returns distribution. However, in the presence of jumps, higher moments are affected and the returns distribution becomes (at least) skewed, strongly affecting the investment decisions.

¹Most large jumps are negative ones. Since the early 80's, 60 percent of the jumps in the Dow Jones Composite Average, larger than 5%, were downwards and only 40 percent were upwards.

As an example of a common durable good we can think about the investing problem of the owner of a house. The decision to sell the house happens when the ratio wealth to house value is below a certain threshold. When there is the possibility of significant downward jumps in the stock market, there is a larger risk that in a given time interval the ratio wealth to house value falls below the critical threshold. Hence, the possibility of large downward jumps tends to anticipate the sale of the durable good, implying an effective increase in the critical threshold for the wealth to house value ratio at which the house is sold.

We followed Damgaard et al (2003) who considered a similar problem in a market where risky assets prices evolve according to a geometric Brownian motion. Given the extensive evidence of non-normality in stock market returns in the financial literature, we consider an extension of this model that includes jumps in stock market prices. Stock market returns distributions are usually left skewed and leptokurtic², which suggests the existence of jumps. There is a wide array of papers in the financial literature that empirically confirm the existence of jumps in stock market returns such as Andersen et al (2002), Eraker et al (2003) and Jarrow and Rosenfeld (1984) who analyze daily time-series of several American stock market indexes and find evidence of jumps in stock market returns. Also, Dunham and Friesen (2007) and Lee and Mykland (2008) among others used ultra-high frequency data on the S&P 500 and concluded that there were jumps in the index returns. Ait-Sahalia et al (2001), Carr and Wu (2003), Jackwerth and Rubinstein (1996) and Pan (2002) studied the jump-risk premia implicit in the S&P 500 options and also found evidence of jumps on the underlying index distribution.

To our knowledge, not many papers have focused on optimal portfolio selection with transaction costs in a stock market with jumps³. Our paper is the first to analyze this problem, where jumps are driven by a Lévy process, and in the presence of both perishable and durable consumption goods. In order to understand the contribution of this paper, we briefly describe the evolution in the literature.

Merton (1969) studied the optimal investment and consumption problem of an individual who consumes only a perishable good with no transaction costs. He assumed that the agent could invest in a riskless asset and a risky asset, whose price process follows a (continuous) geometric Brownian motion. Ignoring transaction costs and other market imperfections, he concluded that a CRRA consumer should invest a constant fraction of his wealth in the risky asset. Obviously, this strategy is not optimal for an investor who faces transaction costs whenever he trades the risky stock since such a strategy would involve continuous trading, and he would face infinite transaction costs. Since Merton (1969), a vast number of papers focused in the optimal consumption and portfolio selection of a consumer that faces transaction costs and/or other market imperfections. Among others Davis and Norman (1990) and Shreve and Soner (1994) studied the optimal portfolio allocation problem of an infinitely lived investor who faces proportional transaction costs when he trades the only risky asset available in the economy. They showed that it is optimal for the investor not to trade continuously the risky asset: there is a wedge shaped no-trading region. Whenever

²See, for example, Andersen et al (2002).

³Benth (2002) and Framstad et al (1999) are two exceptions.

the risky asset investment becomes sufficiently low (high) relative to the riskless asset investment, the investor buys (sells) the risky asset in order to return to the no-trading region limit. Akian et al (1996) extended the previous works with proportional transaction costs by considering that the individual can invest in n risky assets. They concluded that, almost surely, the investor never trades more than one risky asset simultaneously. Liu (2004) considered the problem of investing in n risky assets, but considering also fixed transaction costs and shows that, if risky assets are uncorrelated, the optimal investment policy is to keep the dollar amount invested in each risky asset between two constant levels. Whenever either of these bounds is reached, the agent trades to the corresponding optimal targets. Chellathurai and Draviam (2005), Chellathurai and Draviam (2007), Dai and Fahuai (2009), Liu and Loewenstein (2002), Zakamouline (2005b) and Zakamouline (2002) considered the optimal investment problem of a finite horizon individual, under several specifications for the transaction costs, and concluded that the no-trading region widens as the horizon gets shorter. There is also an extensive literature in the related topic of pricing derivatives in markets with transaction costs. Using similar methodology, several authors such as Barles and Soner (1998), Davis et al (1993), Monoyos (2004), Subramanian (2001) and Zakamouline (2005a) have used the utility indifference price method to determine the reservation buying and selling prices of options, under different specifications for transaction costs.

In our paper we use very similar techniques to measure the impact of jumps in the financial markets together with the simultaneous consumption of durable and perishable goods. The importance of considering both goods at the same time is that it allows for analysing the impact of different transaction cost structures in the presence of the financial market prices' discontinuities. As we show in the remaining of the paper, the combination of these effects imply in a quite subtle investment strategy that has been over-regarded in the literature. Not only the presence of jumps affects the investment strategy in the stock market as one would expect, but also affects the shape of the no-trading region for the durable consumption good. The shape of this region can be shown to be highly driven by the jump modelling and the transaction cost structure.

This paper is organized as follows. The next section presents the basic model; section 3 solves the model in the presence of jumps but in the absence of transaction costs; section 4 presents the solution to the problem with transaction costs; section 5 characterizes the optimal investment and consumption strategies when transaction costs and jumps co-exist; section 6 presents a simulation with parameters that are standard in the literature in order to understand the magnitude and the impact of the effects described in this paper; section 7 develops a sensitivity analysis of the parameters of our numerical experiment; section 8 finally concludes the paper.

2 The model

We consider an economy with two kinds of consumption goods, a perishable good and a durable one, and with two financial assets, a riskless bond B and a risky stock S . The price processes are defined taking the durable good price as the numeraire. The bond B pays a constant, continuously compound interest rate r . The stock S pays no dividend, with a *cadlag* price process that follows a geometric Lévy process

$$dS_t = \mu S_t dt + \sigma S_t dw_{1t} + S_{t-} \int_{-1}^L \eta \tilde{N}(dt, d\eta)$$

where $\mu > r$ and $\sigma > 0$, $L > 0$ are constants, w_{1t} is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and

$$\tilde{N}(t, A) = N(t, A) - tq(A); \quad t \geq 0, \quad A \in B(-1, L)$$

is the compensator of a homogenous Poisson random measure $N(t, A)$ on $\mathbb{R}^+ \times B(-1, L)$ with intensity measure $E[N(t, A)] = tq(A)$, where q is the Lévy measure associated to N and $B(-1, L)$ denotes the Borel σ -algebra on $(-1, L)$. We assume that

$$\|q\| \equiv q((-1, L)) < \infty$$

Note that, since we assume that jump sizes are always greater than -1 , the risky asset price process remains positive for all $t \geq 0$ *a.s.*.

We assume, as in Damgaard et al (2003), that the unit price of the durable good- P_t - follows a geometric Brownian motion

$$dP_t = P_t [\mu_P dt + \sigma_{P_1} dw_{1t} + \sigma_{P_2} dw_{2t}]$$

where $\mu_P > 0$, $\sigma_{P_1} > 0$ and $\sigma_{P_2} > 0$ are constant scalars, and w_{2t} is a Wiener process uncorrelated with w_{1t} . Note that it is impossible to hedge perfectly the risk associated with the durable price process by trading the risky asset, since their price evolution is only partially correlated.

We also assume that the stock of the durable good depreciates at a constant rate δ . This means that at any given time t the stock of the durable good evolves according to the following equation

$$dK_t = -\delta K_t dt$$

if that good is not traded.

Finally, as Damgaard et al (2003) and Grossman and Laroque (1990), we also assume that trading the durable good is costly. More precisely, each time the consumer trades the durable good he must pay a fee proportional to its pre-existing stock. Cuoco and Liu (2000) assume that the transaction cost is proportional to the change in the durable good stock. The first specification is more appropriate for durable goods such as a house or a car, since when a consumer chooses to change the stock of one such durable good, he usually sells it to buy a new one. The later assumption is more appropriate for goods such as furniture, whose stock is typically increased without selling any pre-existing furniture.

The agent must choose a consumption pattern for the perishable good, and a trading strategy for the durable good and the financial assets. Denoting by C_t his consumption rate of the perishable consumption good at time t , we assume that it is a progressively measurable process $C \in \mathcal{L}^1$ where

$$\mathcal{L}^q = \left\{ \{\mathcal{F}_t\} - \text{adapted processes } X : \int_0^T \|X_u(\omega)\| du < \infty \text{ for } P - a.e. \omega \in \Omega, T > 0 \right\}$$

We represent by Θ_{0t} , and Θ_{1t} the amount invested in the riskless asset and the risky asset at time t , respectively. We rule out the possibility of short-selling the risky asset. Therefore, the set of trading strategies consists of the 3-dimensional progressively measurable stochastic processes (Θ_0, Θ_1, K) valued in $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ such that $\Theta_0 \in \mathcal{L}^1$, $\Theta_1 \in \mathcal{L}^2$ and $K \in \mathcal{L}^2$.

We define the agent's wealth X_t as the sum of his investments in the financial assets plus his durable good investments

$$X_t = \Theta_{0t} + \Theta_{1t} + K_t P_t$$

If he follows a strategy (Θ_0, Θ_1, K) , then his wealth evolves according to

$$\begin{aligned} dX_t = & [r(X_t - K_t P_t) + \Theta_{1t}(\mu - r) + (\mu_P - \delta)K_t P_t - C_t] dt + \\ & (\Theta_{1t}\sigma + K_t P_t \sigma_{P1}) dw_{1t} + K_t P_t dw_{2t} + \Theta_t^- \int_{-1}^L \eta \tilde{N}(dt, d\eta) \end{aligned}$$

whenever there is no durable good trade, and

$$X_\tau = X_{\tau-} - \lambda K_{\tau-} P_\tau$$

for every τ where the consumer changes his durable good stock.

We require that the strategies followed by the investor do not lead to bankruptcy. In other words, his wealth must always be higher than the transaction cost he would face if he immediately sold the durable good. Therefore, he must follow a strategy such that, for every $t \geq 0$, the set (X_t, K_t, P_t) must belong *a.s.* to the solvency region

$$S = \{(x, k, p) \in \mathbb{R}_+^3 : x > \lambda k p\}.$$

Let (x, k, p) denote the initial values (X_0, K_0, P_0) . It is natural to assume that if $(x, k, p) \in S$, then *a.s.* for every t , then $(X_t, K_t, P_t) \in S$. Knowing that the risky asset price process may jump, we must strengthen the previous condition by requiring that the trading strategy (Θ_1, K, C) is such that $(X_{t-} - \Theta_1 \eta, K_t, P_t)$ belongs to S , for every $t \geq 0$, and $\eta \in (-1, L)$. We denote the set of admissible strategies (those that satisfy the previous conditions) by $A(x, k, p)$. We will assume that this set is non-empty.

We consider a consumer who maximizes an infinite horizon time-separable utility function, that depends both on his perishable consumption flow and durable good stock

$$\int_0^\infty e^{-\rho t} U(C_t, K_t) dt$$

where $\rho > 0$ is a constant scalar representing the consumer's time preference. We assume that his instantaneous utility function is of the multiplicatively separable isoelastic form

$$U(c, k) = \frac{(c^\beta k^{1-\beta})^{1-\gamma}}{1-\gamma}$$

where β and $\gamma \in (0, 1)$. If the consumer follows the admissible strategy (Θ_1, K, C) his intertemporal expected utility will be

$$J^{\Theta_1, K, C}(x, k, p) = E \left[\int_0^\infty e^{-\rho t} U(C_t, K_t) dt \right]$$

The objective of this agent is to choose an admissible strategy $(\Theta_1, K, C) \in A(x, k, p)$ that maximizes the expected value of his intertemporal utility function, given his initial endowment and durable good price (x, k, p)

$$V(x, k, p) = \sup_{(\Theta_1, K, C) \in A(x, k, p)} J^{\Theta_1, K, C}(x, k, p)$$

3 A semi-explicit solution for the no transaction costs problem

In this section we provide semi-explicit expressions for the optimal consumption flow of the perishable good and the optimal trading strategies for the financial assets and the durable good. Our result generalizes Framstad et al (1999) who considered the optimal consumption and portfolio selection of an agent who consumes only a perishable consumption good, in a Lévy-driven financial asset market. It also extends the result of Damgaard et al (2003), who analyzed the same problem in a market with both a perishable and a durable consumption good, but assuming that the risky financial assets prices follow a geometric Brownian motion (without jumps).

In this section we assume a framework where trading in the durable good involves no cost. Therefore, the agent's optimal consumption and portfolio strategies are independent of the composition of his wealth (the fraction invested in the risky asset and in the durable good). Thus, we can eliminate our state variable K and obtain the optimal portfolio and consumption as a function of the two remaining state variables (X, P) . The following theorem characterizes the optimal solution.

Theorem 1 *Suppose that*

$$\lim_{R \rightarrow \infty} E[e^{-\rho T_R} V(X_{T_R}, P_{T_R})] = 0$$

where $T_R = \min(R, \inf\{|X_t^| \geq R\})$ and X_t^* is the wealth process when the consumer chooses the optimal strategy. Then the value function is given by*

$$V(x, p) = \frac{1}{1-\gamma} \alpha_v p^{-(1-\beta)(1-\gamma)} x^{(1-\gamma)}$$

and the optimal controls are $C_t = \alpha_c X_t^*$, $K_t = \alpha_k X_t^*/P_t$ and $\Theta_{1t} = \alpha_\theta X_t^*$, where

$$\alpha_k = \frac{1}{\gamma \sigma \sigma_{P1}} \left[-\gamma \alpha_\theta \sigma^2 + \mu - r - (1 - \beta)(1 - \gamma) \sigma \sigma_{P1} + \int_{-1}^L \eta [(1 + \alpha_\theta \eta)^{-\gamma} - 1] dq(\eta) \right],$$

$$\alpha_c = \frac{\beta}{1 - \beta} \alpha_k [(1 - \beta)(1 - \gamma)(\sigma_{P1}^2 + \sigma_{P2}^2) + \gamma(\alpha_\theta \sigma \sigma_{P1} + \alpha_k^2(\sigma_{P1}^2 + \sigma_{P2}^2)) + r - \mu_P + \delta],$$

and

$$\alpha_v = \beta \alpha_c^{\beta(1-\gamma)-1} \alpha_k^{(1-\gamma)(1-\beta)}$$

where α_θ is the solution to the equation given in the appendix.

The optimal strategy can be seen to consist of three components: the consumption of the perishable good C_t , the value $P_t K_t$ invested in the durable good and the value Θ_{1t} invested in the risky stock are constant fractions of the total wealth.

In the above solution the presence of jumps is reflected by the integral $\int_{-1}^L \eta [(1 + \alpha_\theta \eta)^{-\gamma} - 1] dq(\eta) < 0$ in the expression for α_k . Therefore, for a given fraction of wealth invested in the risky asset α_θ , the presence of jumps lowers α_k and, as a consequence, lowers the optimal durable good consumption K_t . Likewise, the presence of jumps lowers α_c due to the influence of α_k , and reduces the consumption C_t of perishable good. Hence, overall the consumption is lower than in the model by Damgaard et al (2003), where no jumps were considered in the risky asset price process.

4 Solution to the transaction costs problem

In this section we consider the optimal portfolio selection and consumption policies for an agent who faces positive transaction costs $\lambda > 0$. In this framework we cannot find an explicit solution for the consumer's problem. We characterize the optimal solution and then we find the optimal strategies using numerical simulations.

We show that, just as in Damgaard et al (2003), the existence of fixed transaction costs implies that the durable good will be traded at most at a countable number of times. They have shown that whenever the consumer buys or sells the durable good, the ratio between a consumer's wealth and his durable good stock (prior to the transaction) is constant. That is, the consumer trades the durable good whenever this ratio reaches a given threshold.

We also characterize the stopping times at which the consumer trades the durable good but considering the presence of jumps. Unlike the model studied by the former authors, the ratio between an agent's wealth and his durable good stock immediately before the trade is not necessarily constant due to the existence of jumps in the process driving risky asset prices.

In order to proceed we first show some important properties of the value function, namely its boundness, monotonicity, continuity and homogeneity. First, we assume that the value function is finite and satisfies the dynamic programming equation. More formally we make the following

Conjecture 2 *For all (x, k, p) belonging to the solvency region, an optimal policy exists, and the value function is finite and satisfies the dynamic programming principle. Therefore for all the*

stopping times τ ,

$$V(x, k, p) = \sup_{(\Theta_1, K, P) \in A(x, k, p)} E \left[\int_0^\tau e^{-\rho t} U(C_t, K_t) dt + e^{-\rho \tau} V(X_\tau, K_\tau, P_\tau) \right] \quad (1)$$

where $A(x, k, p)$ represents the space of admissible controls for the initial endowment and durable good price (x, k, p) .

Note that for every t with $0 \leq t \leq \tau$, the investment in the risky financial asset must be such that a jump does not cause insolvency. That is, for every t with $0 \leq t \leq \tau$, $(X_{t-} - \theta_1 \eta, K_t, P_t)$ must belong to the solvency region *a.s.*, for every $\eta \in (-1, L)$.

For every time t before the first time that the durable good is traded, the durable good stock is given by $K_t = K_0 e^{-\delta t}$. Denoting by τ the first time that the durable good is traded, and applying (1), we get

$$V(x, k, p) = \sup_{(\Theta_1, K, P) \in A(x, k, p)} E \left[\int_0^\tau e^{-\rho t} U(C_t, k e^{-\delta t}) dt + e^{-\rho \tau} V(X_{\tau-} - \lambda k e^{-\delta \tau} P_\tau, K_\tau, P_\tau) \right]$$

where K_τ must be such that $(X_{\tau-} - \lambda k e^{-\delta \tau} P_\tau, K_\tau, P_\tau)$ belongs to the solvency region.

The next theorem presents some properties of the value function. We do not provide the proof of this theorem because it follows from Damgaard et al (2003) with only minor modifications.

Theorem 3 *The value function $V(x, k, p)$ satisfies*

1. *For all (x, k, p) belonging to the solvency region*

$$\frac{1}{1-\gamma} \alpha v p^{-(1-\beta)(1-\gamma)} (x - \lambda k p)^{1-\gamma} \leq V(x, k, p) \leq \frac{1}{1-\gamma} \alpha v p^{-(1-\beta)(1-\gamma)} x^{1-\gamma} \quad (2)$$

where α_v is given in Theorem 1, and

$$\underline{\alpha} = \frac{\beta^{\beta(1-\gamma)} (1-\beta)^{(1-\beta)(1-\gamma)} r^{\beta(1-\gamma)}}{\rho + \delta (1-\beta) (1-\gamma)}.$$

2. *For each $(k, p) \in \mathbb{R}_+^2$, $V(x, k, p)$ is strictly increasing and concave in x on the solvency region.*
3. *For each $(k, p) \in \mathbb{R}_+^2$, $V(x, k, p)$ is continuous in x on the solvency region.*
4. *$V(x, k, p)$ is homogeneous of degree $1-\gamma$ in (x, k) and of degree $\beta(1-\gamma)$ in (x, p) for all (x, k, p) belonging to the solvency region.*

Now, we will use the homogeneity of the value function to reduce the dimensionality of the problem. Note that

$$V(x, k, p) = k^{1-\gamma} p^{\beta(1-\gamma)} v(x/(kp)) \quad (3)$$

for every (x, k, p) belonging to the solvency region where $v(z) = V(z, 1, 1)$, and the solvency region is the closure of the set $z \in (\lambda, \infty)$. If $z = \lambda$, the agent must sell his entire stock of the durable good to avoid insolvency. Therefore $v(\lambda) = 0$. Note also that because of Theorem 3, item 1, the transformed value function is also bounded, with

$$\frac{1}{1-\gamma} \alpha (z - \lambda)^{1-\gamma} \leq v(z) \leq \frac{1}{1-\gamma} \alpha_v z^{1-\gamma}$$

Let $Z_t = X_t / (K_t P_t)$ denote the transformed wealth process and $\hat{C}_t = C_t / (K_t P_t)$ and $\hat{\Theta}_{1t} = \Theta_{1t} / (K_t P_t)$ denote the transformed controls for perishable consumption and risky asset investment, respectively. Note that the condition that we presented before, and that must be satisfied by Θ_{1t} to avoid bankruptcy, becomes, in the transformed model

$$\hat{\Theta}_{1t} : \left\{ Z_t - \hat{\Theta}_{1t} \eta \geq \lambda \right\} \quad \text{for every } t : t \geq 0, \eta \in (-1, L), \text{ a.s.}$$

Using the dynamic programming principle and (3) we obtain the following equation

$$p^{\beta(1-\gamma)} v(z) = \sup_{(\hat{\Theta}_1, \hat{C}) \in A(z)} E \left\{ \frac{1}{1-\gamma} \int_0^\tau e^{-\bar{\rho}t} \hat{C}_t^{\beta(1-\gamma)} P_t^{\beta(1-\gamma)} dt + e^{-\rho\tau} P_\tau^{\beta(1-\gamma)} \frac{(Z_{\tau^-} - \lambda)^{1-\gamma}}{1-\gamma} M \right\}$$

where $p = P_0$, $z = Z_0$, $\bar{\rho} = \rho + \delta(1-\gamma)$, $A(z)$ is the set of admissible controls for initial transformed wealth z , τ is the first time when the agent trades the durable good,

$$M = \sup_{K_\tau \leq k e^{-\delta\tau} (Z_{\tau^-} - \lambda) / \lambda} (1-\gamma) \left(\frac{k e^{-\delta\tau} (Z_{\tau^-} - \lambda)}{K_\tau} \right)^{\gamma-1} v \left(\frac{k e^{-\delta\tau} (Z_{\tau^-} - \lambda)}{K_\tau} \right) \quad (4)$$

and $k = K_0$.

This framework combines a stochastic control problem and an impulse control problem for a jump diffusion. Its solution may be approximated by the solution of a stochastic control problem combined with the solution of an optimal stopping problem⁴, where the terminal value function is given by $f(z) = \frac{(z-\lambda)}{1-\gamma} M$. Note that we can recover the optimal durable good stock immediately after a transaction from the following equality

$$K_\tau = \frac{K_{\tau^-} (Z_{\tau^-} - \lambda)}{z^*}$$

where $z^* = \arg \max_{z \geq \lambda} z^{\gamma-1} v(z)$.

The Hamilton-Jacobi-Bellman (HJB) equation for the problem defined above is

$$0 = \max \left\{ H(z, v, v', v''), \frac{(z-\lambda)^{1-\gamma}}{1-\gamma} M - v(z) \right\} \quad (5)$$

where

⁴See, for example, Oksendal and Sulem, 2004.

$$H(z, v, v', v'') = \sup_{\hat{\theta}_1, \hat{c} \in A(z)} \left\{ \mathcal{L}^{\hat{\theta}_1, \hat{c}} v(z) + \frac{1}{1-\gamma} \hat{c}^{\beta(1-\gamma)} - \bar{\rho} v(z) \right\} \quad (6)$$

and

$$\begin{aligned} \mathcal{L}^{\hat{\theta}_1, \hat{c}} v(z) &= v(z) \beta(1-\gamma) \left[\mu_P - \frac{1}{2} (1-\beta(1-\gamma)) (\sigma_{P_1}^2 + \sigma_{P_2}^2) \right] + \\ &v'(z) \left[\frac{r(z-1) + \hat{\theta}_1(\mu-r) - \hat{c} + \delta z - \mu_P(z-1) - \delta}{(1-\beta(1-\gamma))} \left[(z-1)(\sigma_{P_1}^2 + \sigma_{P_2}^2) + \hat{\theta}_1 \sigma \sigma_{P_1} \right] \right] + \\ &v''(z) \left[\frac{1}{2} (\sigma_{P_1}^2 + \sigma_{P_2}^2) (z-1)^2 + \hat{\theta}_1 \sigma \sigma_{P_1} (1-z) + \frac{1}{2} \hat{\theta}_1^2 \sigma^2 \right] + \\ &\int_{-1}^L \left[v(z + \hat{\theta}_1 \eta) - v(z) - v'(z) \hat{\theta}_1 \eta \right] dq(\eta) \end{aligned}$$

Unfortunately, due to the presence of transaction costs, we cannot guarantee that the transformed value function is sufficiently smooth, *i.e.*, the derivatives $v'(z)$ and $v''(z)$ may be not well-defined. In that case it would be difficult to characterize the function $H(z, v, v', v'')$ and to find a solution that satisfies the HJB equation in the classical sense above. The typical solution for these cases is to approach the HJB problem by a similar-class problem with a so-called viscosity solution⁵. It can be shown that all solutions of an HJB problem are solutions of a viscosity problem, but the reverse is not necessarily true. In that sense, the problem solved by the viscosity solution can be seen as a generalization of the HJB problem. We use this concept of viscosity solutions to show that the above HJB problem has a solution in this weaker sense. We will also prove that this solution is continuous on all the solvency region. Next, theorem 4 shows that, under certain conditions, the viscosity solution is unique. Proofs are presented in the Appendix.

Theorem 4 *If $\rho > (1-\gamma) \left(\beta \mu_P - 1/2 \beta (1-\beta(1-\gamma)) (\sigma_{P_1} + \sigma_{P_2})^2 - \delta \right)$ then the function $v(z)$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation for the transformed problem and v is continuous in $[\lambda, \infty)$.*

Theorem 5 *Let us assume that*

1. $\rho > (1-\gamma) \left(\beta \mu_P - 1/2 \beta (1-\beta(1-\gamma)) (\sigma_{P_1} + \sigma_{P_2})^2 - \delta \right)$;
2. \underline{v} and \bar{v} are a continuous subsolution, and supersolution, respectively, with sublinear growth in $[\lambda, \infty)$, and with $\underline{v}(z) \geq \frac{M}{1-\gamma} (z-\lambda)^{1-\gamma}$;
3. $\underline{v}(\lambda) = \bar{v}(\lambda) = 0$;
4. $\tilde{\theta} = \hat{\theta}$, where $\tilde{\theta}$ and $\hat{\theta}$ are given in the appendix.

Then $\underline{v}(z) \leq \bar{v}(z)$.

⁵See, for example, Crandall et al (1992).

Unicity of the viscosity solution follows from the Theorem above together with the fact that $\underline{v}(z) \geq \bar{v}(z)$ by definition, and also because $v(z)$ has to be in between those bounding functions. Thus $v(z) = \underline{v}(z) = \bar{v}(z)$.

5 Characterization of the optimal consumption and trading strategies

In this section we show that, under specific conditions, the solvency region can be divided in three zones. In the first one it is optimal to sell the durable good, in the second the agent does not trade the durable consumption good, and in the last one the optimal strategy is to buy the durable good. As before, let z denote the ratio between the total wealth and the value of the stock of durable goods. The decision to sell the durable good happens when the ratio z is below a certain threshold. When there is the possibility of significant downward jumps in the stock market, there is a larger risk that in a given time interval the ratio z falls below the critical threshold. Hence, the possibility of large downward jumps tends to anticipate the sale of the durable good, implying an effective increase in the critical threshold for z at which the good is sold. The next theorem shows that the no-trading region is the set

$$N = \{z > \lambda : v(z) > f(z)\}$$

where $f(z) = \frac{(z-\lambda)^{1-\gamma}}{1-\gamma}M$ is the stopping reward function, where M reflects the Value function just after the transaction of the durable good, just as defined in section 4. We will also show that the no-trading region is an interval, delimited by \bar{z} and \underline{z} , where $\bar{z} > \underline{z} > \lambda$. Whenever the agent is outside the no-trading region, transaction of the durable good takes place and the value of the state variable z immediately after the trade is reset to z^* located in the no-trading region N .

Theorem 6 *If $H(z, f, f', f'')$ is increasing in the second argument and the set $\{z : H(z, f, f', f'') > 0\}$ is an interval, then there exist numbers \bar{z} and \underline{z} with $\bar{z} > \underline{z} > \lambda$ such that $N = (\underline{z}, \bar{z})$, and z^* , defined above belongs to the no-trading region.*

The proof of theorem 5 follows from Damgaard et al (2003) with slight modifications.

The following figure shows the buying, selling and no trading region of the risky asset in the (x, kp) space.

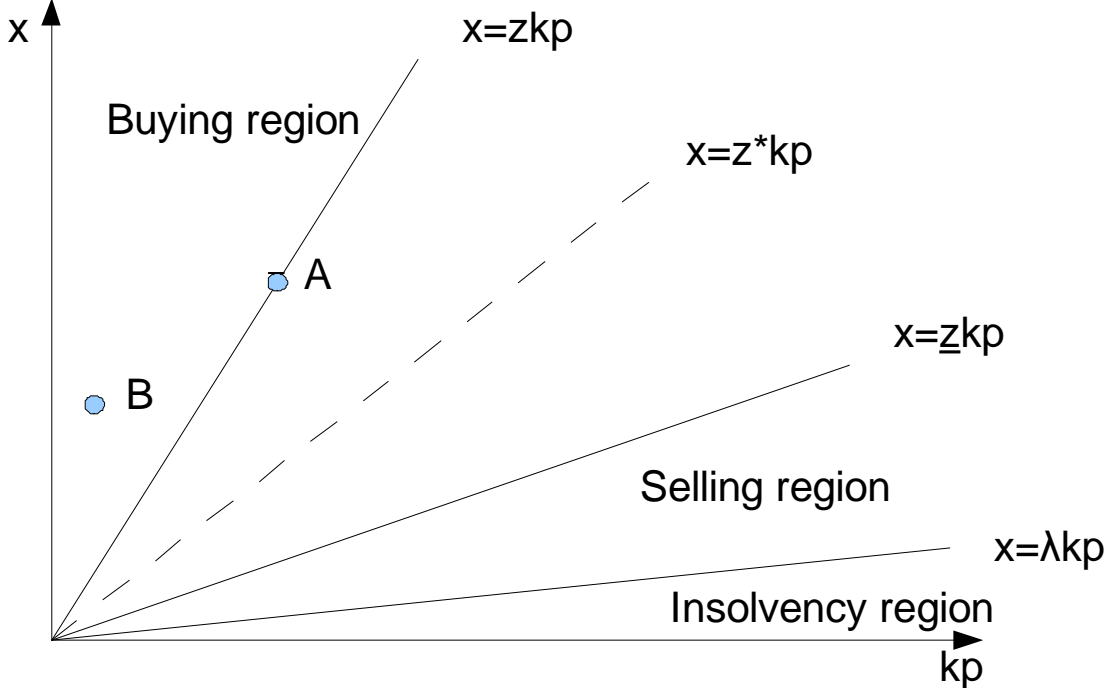


Figure 1: Buying, selling and no-trading regions

The no-trading region corresponds to the cone

$$N = \left\{ (x, k, p) : \underline{z} < \frac{x}{kp} < \bar{z} \right\}$$

When $\frac{x}{kp}$ attains the lower boundary of the no-trading region, the durable good stock becomes too high relative to the agent's total wealth. Therefore, he sells the durable good, and the state variable z value after the trade will equal z^* . If z gets out of the no-trading region through its upper boundary, the durable good stock becomes too low, and the agent buys the durable good until $z = z^*$. Note that the optimal strategy involves infrequent trading: the agent executes an initial durable good trade if z lies outside the no-trading zone, and then he only trades this good again once z attains one of the no-trading region boundaries.

It can be shown that the slope of the straight line representing the transaction in the (x, kp) space is $\lambda z^* / (z^* - z^- + \lambda)$, where z^- represents the state variable value immediately before the transaction. In Damgaard et al (2003) this slope equals the constant $\lambda z^* / (z^* - \bar{z} + \lambda)$ when the agent buys the durable good and $\lambda z^* / (z^* - \underline{z} + \lambda)$ when he sells the durable good. z is continuous, and therefore, the agent trades the durable good at the stopping time $\{\tau : z_\tau = \underline{z} \text{ or } z_\tau = \bar{z}\}$. Note that, in our model, the risky asset price evolves according to a Lévy process, which implies that the state variable z may jump over any of the no-trading region boundaries. Therefore, he may trade the risky asset at a point such as A in the graph above, if the Brownian motion part drives the value of z to one of the boundaries, but he may also trade the durable good at a point such as B , if a jump in the risky asset price causes z to jump over any of the boundaries.

In the last part of this section we present the optimal perishable good consumption and the optimal risky asset investment strategy. Using equation (6) we have

$$c^* = \left(\frac{v'(z)}{\beta} \right)^{-1/(1-\beta(1-\gamma))}$$

and θ^* is the solution to

$$\begin{aligned} 0 = & v'(z)(\mu - r - (1 - \gamma(1 - \gamma))\sigma\sigma_{P_1}) + v''(z)(\theta\sigma^2 - (z - 1)\sigma\sigma_{P_1}) \\ & + \int_{-1}^L [v'(z + \theta\eta)\eta - v'(z)\eta] dq(\eta) \end{aligned} \quad (7)$$

Note that in this equation there are two extra terms that are not present in the solution for a mean-variance optimizing investor in a Brownian motion context. The term $(z - 1)\sigma\sigma_{P_1}$ is a hedging term that is related to the correlation between the risky asset price and the durable good price, and $\int_{-1}^L [v'(z + \theta\eta)\eta - v'(z)\eta] dq(\eta)$ derives from the existence of jumps in the risky asset price.

6 Numerical results

In this section we present the results of the numerical simulations for the transaction costs problems. A detailed description of the algorithm used is provided in the appendix B. In the numerical simulations we used the following parameter values for the baseline scenario

PREFERENCES- We followed Damgaard et al (2003) and assumed $\beta = 0,5$, $\gamma = 0,5$ and $\rho = 0,2$.

ASSET PRICE PROCESSES- We assume that the riskless rate equals 5%. Regarding the risky asset price process, we consider $\mu = 12\%$, and $\sigma = 25\%$. We also assume that the jump process distribution is degenerate- there is a 20% probability of a -10% jump each year.

DURABLE PRICE PROCESS- Once again we follow Damgaard et al (2003) and assume that $\mu_P = 7\%$, the standard deviation of the durable good price equals 12%, and the correlation coefficient between the durable good price and the risky asset price equals 0,2. The durable good depreciates at a 4% rate a year. The durable good transaction cost equals 5% of its pre-existing stock value.

In the following figures we will compare our baseline scenario, that involves jumps in the risky asset price process, with an alternative framework, in which there are no risky asset price jumps. Figure 2 displays the difference between the value function and its value if a durable asset transaction were performed $\left(v(z) - \frac{M}{1-\gamma}(z - \lambda)^{1-\gamma} \right)$. As in Damgaard et al (2003) and Grossman and Laroque (1990) the value function is less concave close to the boundaries of the no-trading region than at its middle. Regarding the difference between our scenario with jumps in the risky asset price and the alternative scenario with no jumps, we observe that with the existence of jumps the no-trading region becomes narrower (with jumps $\underline{z} = 0,15$ and $\bar{z} = 2,6$, and without jumps $\underline{z} = 0,15$ and $\bar{z} = 2,72$). We assumed that jumps equals -10% of the risky asset price. Therefore,

jumps cause two effects on the risky asset price distribution: (i) its variance increases and (ii) the distribution becomes skewed to the left. The first factor leads to a decrease in the size of the no-trading region (an increase in \underline{z} and a decrease in \bar{z}), but the second one causes a decrease in \underline{z} —the agent decreases \underline{z} in order to avoid that a jumps generate too many durable good transactions.

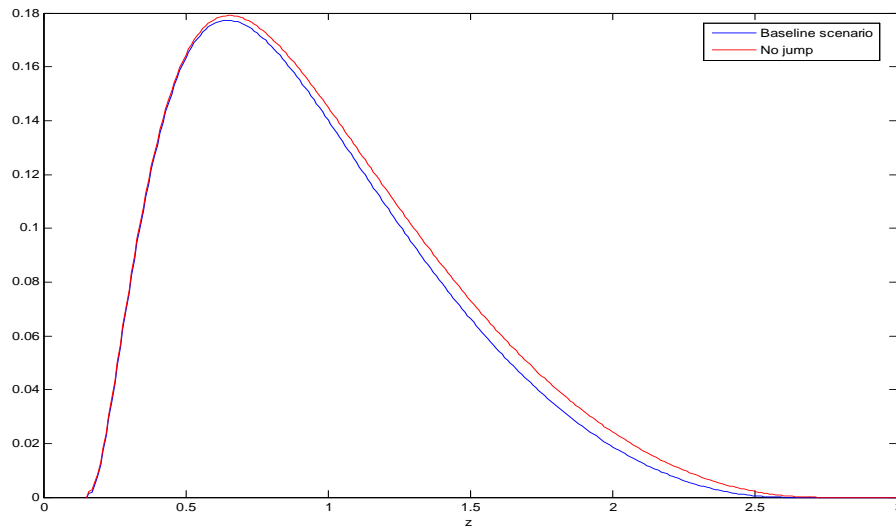


Figure 2: No-trading region boundaries

Figure 3 displays the agent's relative risk aversion. Relative risk aversion is only slightly affected by the existence of jumps. Its pattern is very similar in the scenarios with jumps and without jumps. It is low near the boundaries of the no-trading region because the utility loss from trading the durable good is low near \underline{z} and \bar{z} (see figure 1), and it increases as z approaches the middle of the no-trading region. The agent behaves in a more risk averse manner z^* because he wants to avoid the cost of trading the durable good. Near the boundaries, the loss from these transaction cost is partially compensated by the benefit derived from the change to the optimal wealth to durable good ratio z^* .

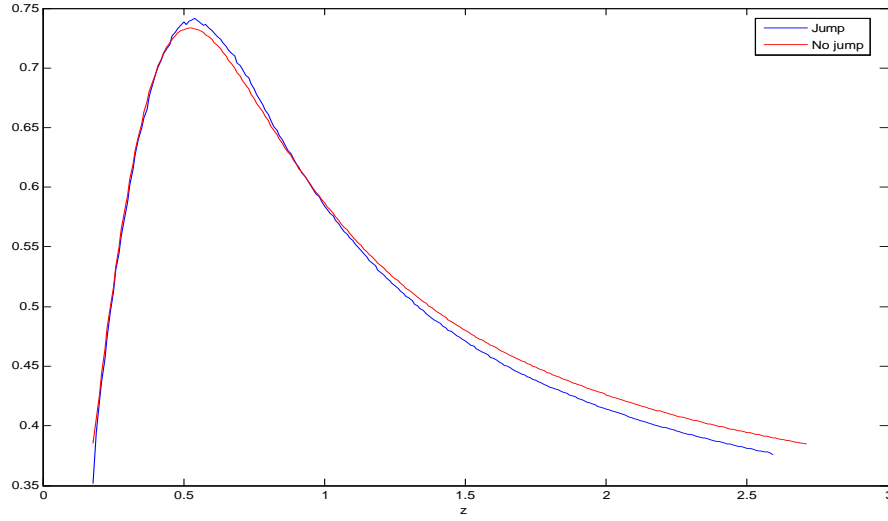


Figure 3: Relative risk aversion

Figure 4 shows the fraction of wealth invested in the risky asset⁶.

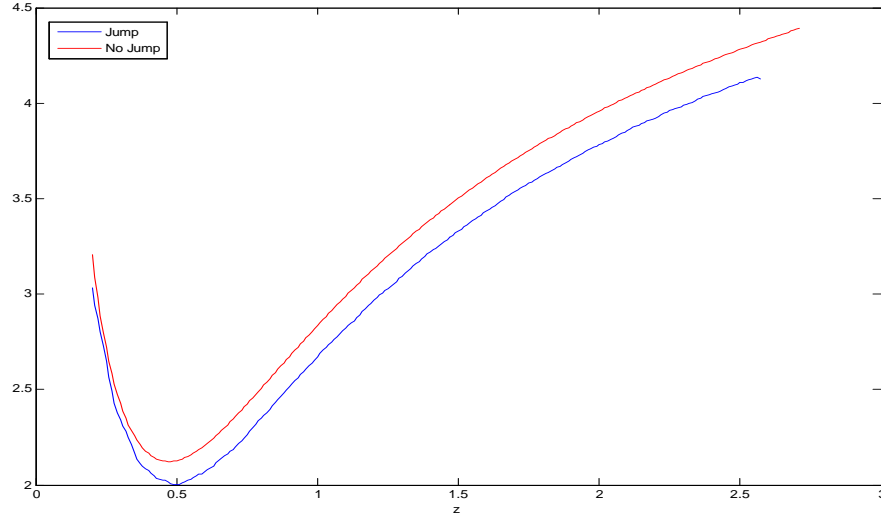


Figure 4: Fraction of wealth invested in the stock market

The optimal portfolio depends on three complementary factors (i) the mean-variance ratio of the risky asset price process, (ii) the correlation between the risky asset price process and the durable good price process and (iii) the existence of jumps. The first component varies positively with the mean-variance ratio, and decreases as the agent's relative risk aversion increases. Therefore, this term is higher near the boundaries of the no trading region and lower at the middle. The second term is negative, because the positive correlation between the risky asset price process and the durable good price generates a negative hedging demand. The absolute value of this hedging

⁶Note that $\hat{\theta}_z = \frac{\theta}{x}$. Therefore we can easily calculate the fraction of wealth invested in the risky asset from the variables of the transformed model.

demand is decreasing in z , because for lower values of z the durable good stock has a higher weight in the agent's wealth. Finally, jumps increase the risky asset price variance and have a negative effect on the risky asset allocation. This effect is evident in the following figure that compares the optimal risky asset allocation for agents that face risky asset price processes with and without jumps⁷.

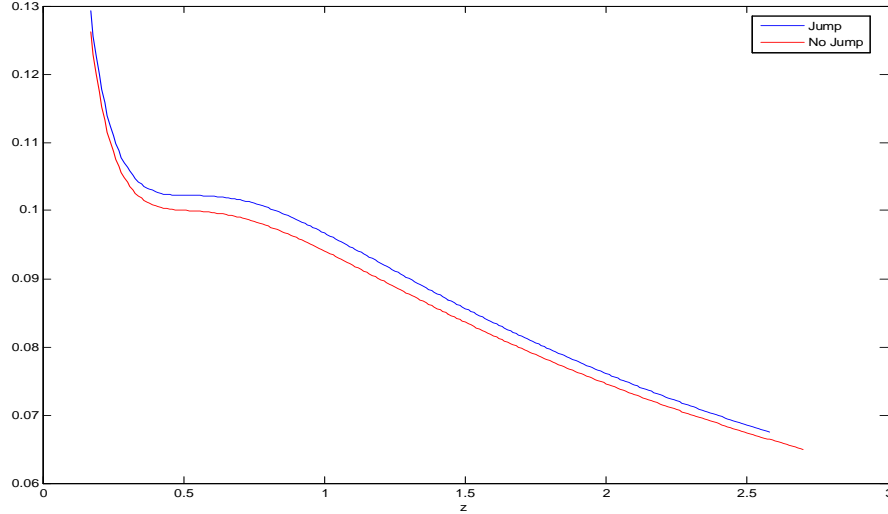


Figure 5: Optimal propensity to consume

Figure 5 displays the optimal propensity to consume as a function of z . As shown in Damgaard et al (2003) the optimal propensity to consume is increasing in z whenever the relative risk aversion with respect to wealth changes is higher than the "relative risk aversion" with respect to consumption changes (i.e. $-zv''(z)/v'(z) > -c\frac{\partial^2 U}{\partial c^2}/\frac{\partial U}{\partial c} = 1 - \beta(1 - \gamma)$). In our base scenario, $1 - \beta(1 - \gamma) = 0,75$. Therefore, the optimal propensity to consume is always decreasing (see figures 2 and 4). Note also that an agent that faces a risky asset price with jump consumes a smaller fraction of wealth than an agent that can invest in a risky asset with no price jumps, for each z . This pattern is a consequence of the fact that jumps increase the variance of the risky asset price, and consequently, this asset becomes a less effective vehicle to transfer consumption to the future, and the agent chooses to increase his present consumption.

7 Parameter sensitivity

In this section we analyze the sensitivity of the agent's choices to several parameters of our model. In figure 6 we show the effect of changing the durable good transaction cost on the no trading region. The figure shows that as λ increases, the no-trading region widens. The agent chooses to increase the size of the no trading region to avoid transacting the durable good too often. Note that he trades the durable good when the cost of trading equals the benefit derived from the durable

⁷Note that the existence of jumps makes risky asset investment a poorer hedge against durable price change, *ceteris paribus*.

stock rebalancing. Then, as the transaction cost increases, the agent is willing to accept higher imbalances in his portfolio.

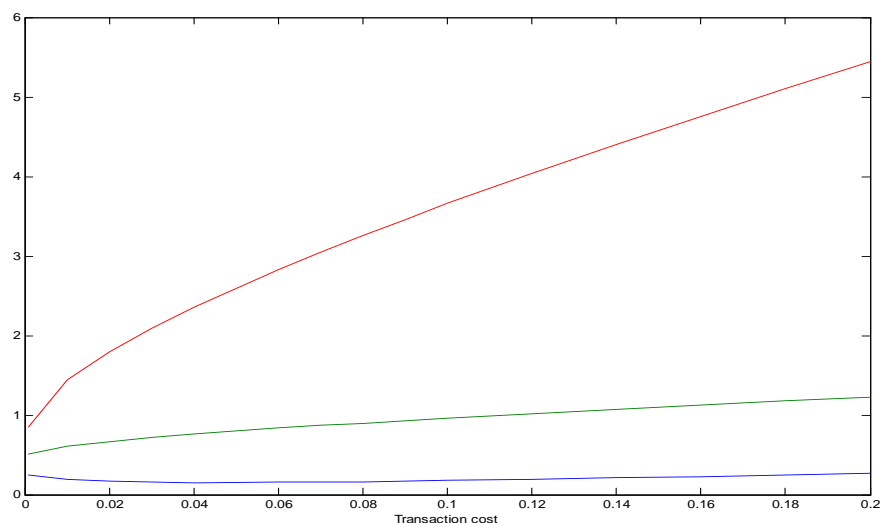


Figure 6: No-trading region

Figure 6 also shows that the optimal wealth to durable good ratio after a transaction is increasing in the transaction costs. When the agent decides on the stock of the durable good that he owns after a trade, he must weigh two factors: (i) a higher durable good stock implies that the consumer must pay a higher transaction cost the next time he trades it and, therefore, he should hold a lower durable good stock (ii) the fact that the no trading region is wider implies that it takes longer until the next durable good trade, which implies that he should hold a higher durable good stock to compensate its depreciation. As in Damgaard et al (2003) the first effect prevails, and z^* is increasing in λ .

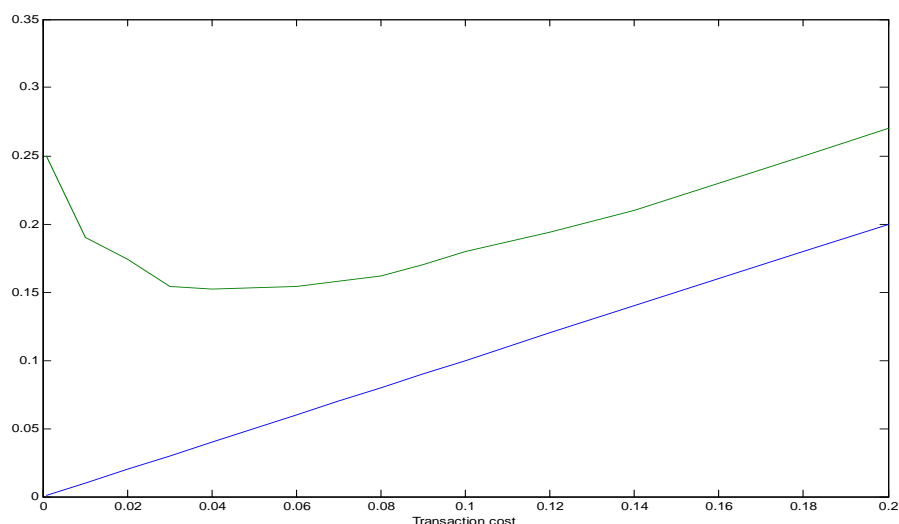


Figure 7: Lower boundary of the no-trading region

The upper boundary of the no-trading region is more sensitive to the transaction costs than the lower boundary because, as λ increases, the solvency region gets smaller. The lower boundary of the solvency region imposes a lower limit on the value of \underline{z} . Figure 7 shows that as the transaction cost increases \underline{z} gets closer to this lower boundary.

The following table provides an analysis of the sensitivity of the no-trading region (\underline{z} and \bar{z}), and the optimal durable good to wealth ratio ($1/z^*$), the propensity to consume (c^*), the fraction of wealth invested in the risk asset $(\theta/x)^*$, and the relative risk aversion, all calculated at the value of z immediately after a durable good transaction.

The parameter β controls the relative weight of perishable consumption to durable consumption in the instantaneous utility function. When β decreases (increases) the weight of durable consumption increases (decreases) and the agent increases (decreases) his durable good stock (z^*). The no-trading region becomes narrower as β decreases because, as the weight of the durable good on the instantaneous utility function becomes larger, potential imbalances in his durable good stock have a higher impact on his utility. Perishable good consumption rate evolves in the opposite way, that is, it increases as β becomes higher. The fraction of wealth invested in the risky asset also depends positively on β . This pattern can be explained by the decrease in the relative risk aversion at the level z^* , and also by the decrease in the absolute value of the hedging demand. Higher β imply a lower durable good stock and, consequently, a less negative risky asset hedging demand.

	\underline{z}	\bar{z}	$1/z^*$	RRA	c^*	θ/x
Benchmark	0,15	2,6	1,25	0,66	0,10	2,35
$\beta = 0,3$	0,12	2,04	1,61	0,71	0,059	2,15
$\beta = 0,7$	0,22	4,09	0,8	0,59	0,138	2,78
Correlation=0	0,14	2,57	1,32	0,64	0,098	2,52
Correlation=-0,2	0,12	2,56	1,41	0,65	0,093	2,75
Jump=-0,05	0,15	2,71	1,22	0,64	0,98	2,52
Jump=-0,15	0,19	2,44	1,28	0,67	0,104	2,08
Prob.=0,1	0,15	2,66	1,23	0,65	0,099	2,45
Prob.=0,4	0,15	2,52	1,27	0,64	0,103	2,17

The fourth and fifth row of the table display the effect of decreasing the correlation coefficient between the Brownian motion driving the durable good and risky asset price processes. As the correlation becomes negative, the durable good and the risky asset hedge each others price changes. Therefore, the hedging demand becomes positive and the agent increases the fraction of his wealth invested in both the risky asset and the durable good. Note also that, in this case, the risky asset becomes a more effective and less risky vehicle to transfer consumption in to the future and, consequently, the agent decreases current perishable good consumption.

The last rows of the table shows the impact of changing the jump size and the probability of occurrence of a jump in the risky asset price. If the jump becomes more negative, the variance of the risky asset becomes higher, and its distribution becomes more skewed to the left, which implies a decrease in the investment in the risky asset. Also, the fact that the risky asset becomes a less desirable investment opportunity, leads the agent to a higher current consumption. The

increase in the variance of the risky asset generates a decrease in the size of the no-trading region, because wealth allocation imbalances have a higher impact on the utility function. Note that a very negative jump may drive the value of z very close to the lower boundary of the solvency region. Then, in order to avoid exiting the solvency region the consumer increases the lower boundary of the no-trading region.

The consequences of an increase in the probability of a jump are very similar to the impact of an increase in the absolute value of a negative jump- risky asset investment decreases, the no trading region widens and current consumption increases. Though, there is an important difference between the two scenarios: an increase in the probability of jumps has does not change the lower boundary of the no-trading region, unlike in the case where jumps become more negative.

8 Concluding remarks

In this paper we studied the problem of an agent who faces a geometric Lévy stock market price process and consumes both a durable and perishable good. This generates a returns' distribution with a higher variance than the distribution in the no-jump case. This increase in the stock market risk implies a narrower no-trading region, a higher current consumption and a lower investment in the stock market.

We showed that, in the presence of durable good transaction costs, the agent must not trade the durable good continuously, in order to avoid insolvency. The solvency region is divided in three zones: a buying region, a selling region and a no-trading region. Whenever the ratio of wealth-to-durable good value becomes sufficiently low, the agent sells the durable good; whenever that ratio is too high, he buys the good. In Damgaard et al (2003), where no jumps in the stock market are present, the agent only trades the durable good exactly at the boundary between the no-trading and buying region or at the boundary between the no-trading and the selling region. In our framework, a jump in the stock market may cause a sudden change in consumer's wealth and as a consequence, a sudden change in the wealth-to-durable good value ratio. Therefore, the wealth-to-durable good ratio may cross any of the boundaries of the no-trading region after a jump, even if in the previous moment this ratio was stricly inside the no-trading region, leading the consumer to trade the durable good. Hence, on one hand the agent would prefer to shrink the no-trading region in order to reduce the risk of finding himself outside the interval due to the jumps, but on the other hand he perceives a drastic shrinking as increasing the frequency and therefore the costs of transacting the durable good. The trade-off of these two effects determines the optimal shape of the no-trading region.

Also, jumps may be quite asymmetric and the increase in the risk may not be even. In this paper we assumed the existence of negative jumps in the stock market price. This generates a returns' distribution skewed to the left implying that wealth can diminish with a higher probability than it can increase. This means that the wealth-to-durable good ratio is more likely to decrease, thus affecting the threshold that characterizes the selling of the durable good in a different way than it affects the buying threshold. In fact, the buying threshold is determined uniquely by the variance

increase effect whereas the selling threshold is additionally affected by the jumps' occurrence.

9 Appendix A

9.1 Proof of Theorem 1

The value function of the no transaction costs problem is

$$V(x, p) = \sup_{(\Theta_1, K, C) \in A(x, k, p)} E \left[\int_0^\infty e^{-\rho t} U(C_t, K_t) dt \right]$$

The Hamilton-Jacobi-Bellman equation for this problem is

$$\rho V(x, p) = \sup_{\theta_1, c, k \in \mathbb{R}^+} \left\{ \begin{aligned} & \frac{1}{1-\gamma} (c^\beta k^{1-\beta})^{1-\gamma} + (r(x - kp) + \theta_1(\mu - r) + (\mu_P - \delta)kp - c) \frac{\partial V}{\partial x} \\ & + \frac{1}{2} (\theta_1^2 \sigma^2 + k^2 p^2 (\sigma_{P_1}^2 + \sigma_{P_2}^2) + 2\theta_1 k p \sigma \sigma_{P_1}) \frac{\partial^2 V}{\partial x^2} + \mu_P p \frac{\partial V}{\partial p} \\ & + \frac{1}{2} p^2 (\sigma_{P_1}^2 + \sigma_{P_2}^2) \frac{\partial^2 V}{\partial p^2} + p (\sigma_{P_1} (\theta_1 \sigma + k p \sigma_{P_1}) + k p \sigma_{P_2}^2) \frac{\partial^2 V}{\partial x \partial p} \\ & + \int_{-1}^L [V(x + \eta \theta_1, p) - V(x, p) - \frac{\partial V}{\partial x} \theta_1 \eta] dq(\eta) \end{aligned} \right\} \quad (A1)$$

We follow Damgaard et al (2003) and use the homogeneity of the instantaneous utility function to reduce the dimensionality of the problem. Note that, for $\kappa > 0$, (Θ_1, K, C) is admissible with initial wealth x and initial durable price p , if and only if $(\kappa \Theta_1, K, \kappa C)$ is admissible with initial wealth κx and initial durable price κp . Since $U(\kappa C, K) = \kappa^{\beta(1-\gamma)} U(C, K)$ it follows that $V(\kappa x, \kappa p) = \kappa^{\beta(1-\gamma)} V(x, p)$, and in particular $V(x, p) = p^{\beta(1-\gamma)} V(x/p, 1) = p^{\beta(1-\gamma)} v(x/p)$. Substituting this equality in (A1) and simplifying we obtain

$$\begin{aligned} 0 = & -v(y) \left[\rho - \beta(1-\gamma)\mu_P + \frac{1}{2} (\sigma_{P_1}^2 + \sigma_{P_2}^2) \beta(1-\gamma) [1 - \beta(1-\gamma)] \right] \\ & + \sup_{\hat{\theta}_1, k, \hat{c} \in \mathbb{R}^+} \left\{ \begin{aligned} & \frac{1}{2} v''(y) [\hat{\theta}_1^2 \sigma^2 + (\sigma_{P_1}^2 + \sigma_{P_2}^2) (y - k)^2 + 2\hat{\theta}_1 \sigma \sigma_{P_1} (k - y)] \\ & + v'(y) \left[(r + (1 - \beta(1 - \gamma)) (\sigma_{P_1}^2 + \sigma_{P_2}^2) - \mu_P) (y - k) - \delta k + \right. \\ & \quad \left. \hat{\theta}_1 (\mu - r - (1 - \beta(1 - \gamma)) \sigma \sigma_{P_1}) - \hat{c} \right] \\ & + \int_{-1}^L [v(y + \hat{\theta}_1 \eta) - v(y) - v' \hat{\theta}_1 \eta] dq(\eta) + \frac{1}{1-\gamma} (\hat{c}^\beta k^{1-\beta})^{1-\gamma} \end{aligned} \right\} \end{aligned} \quad (A2)$$

where $\hat{c} = c/p$ and $\hat{\theta}_1 = \theta_1/p$. We claim that this equation as a solution of the form

$$v(y) = \frac{1}{1-\gamma} \alpha_v y^{1-\gamma}$$

with the maximizing controls $\hat{c} = \alpha_c y$, $\hat{\theta}_1 = \alpha_\theta y$ and $k = \alpha_k y$. Substituting the previous equation in (A2), we obtain

$$0 = - \left[\rho - \beta (1 - \gamma) \mu_P + \frac{1}{2} \beta (1 - \gamma) (1 - \beta (1 - \gamma)) (\sigma_{P_1}^2 + \sigma_{P_2}^2) \right] + \sup_{\hat{\theta}_1, k, \hat{c} \in \mathbb{R}_+} \left\{ \begin{aligned} & -\frac{1}{2} \gamma \alpha_v y^{-\gamma-1} \left[\hat{\theta}_1^2 \sigma^2 + (\sigma_{P_1}^2 + \sigma_{P_2}^2) (y - k)^2 + 2 \hat{\theta}_1 \sigma \sigma_{P_1} (k - y) \right] \\ & + \alpha_v y^{-\gamma} \left[(r + (1 - \beta (1 - \gamma)) (\sigma_{P_1}^2 + \sigma_{P_2}^2) - \mu_P) (y - k) - \delta k + \right. \\ & \quad \left. \hat{\theta}_1 (\mu - r - (1 - \beta (1 - \gamma)) \sigma \sigma_{P_1}) - \hat{c} \right] \\ & + \int_{-1}^L \left[\frac{\alpha_v}{1-\gamma} (y + \hat{\theta}_1 \eta)^{1-\gamma} - \frac{\alpha_v}{1-\gamma} y^{1-\gamma} - \alpha_v y^{-\gamma} \hat{\theta}_1 \eta \right] dq(\eta) + \frac{1}{1-\gamma} (\hat{c}^\beta k^{1-\beta})^{1-\gamma} \end{aligned} \right\}.$$

Ignoring the positivity constraints, the first order conditions for the maximizing controls are

$$0 = \beta (\alpha_c y)^{\beta(1-\gamma)-1} (\alpha_k y)^{(1-\beta)(1-\gamma)} - \alpha_v y^{-\gamma}$$

$$0 = \gamma \alpha_v y^{-\gamma-1} \left(-\hat{\theta}_1 \sigma^2 + \sigma \sigma_{P_1} \right) (y - k) + \alpha_v y^{-\gamma} (\mu - r - (1 - \beta (1 - \gamma)) \sigma \sigma_{P_1}) + \int_{-1}^{\infty} \left[\alpha_v (y + \hat{\theta}_1 \eta)^{-\gamma} \eta - \alpha_v y^{-\gamma} \eta \right] dq(\eta)$$

$$0 = (1 - \beta) \hat{c}^{\beta(1-\gamma)} k^{(1-\beta)(1-\gamma)-1} + \gamma \alpha_v y^{-\gamma-1} \left[(\sigma_{P_1}^2 + \sigma_{P_2}^2) (y - k) - \hat{\theta}_1 \sigma \sigma_{P_1} \right] - \alpha_v y^{-\gamma} [r + (1 - \beta (1 - \gamma)) (\sigma_{P_1}^2 + \sigma_{P_2}^2) - \mu_P + \delta].$$

Using the first order conditions and the assumed control specification we get

$$\alpha_v = \beta \alpha_c^{\beta(1-\gamma)-1} \alpha_k^{(1-\beta)(1-\gamma)}$$

$$\alpha_c = \frac{\beta}{1-\beta} \alpha_k [(1 - \beta) (1 - \gamma) (\sigma_{P_1}^2 + \sigma_{P_2}^2) + \gamma \alpha_\theta \sigma \sigma_{P_1} + r - \mu_P + \delta + \gamma \alpha_k (\sigma_{P_1}^2 + \sigma_{P_2}^2)]$$

$$\alpha_k = \frac{1}{\gamma \sigma \sigma_{P_1}} \left[-\gamma \alpha_\theta \sigma^2 + \mu - r - (1 - \beta) (1 - \gamma) \sigma \sigma_{P_1} + \int_{-1}^L \eta [(1 + \alpha_\theta)^{-\gamma} - 1] dq(\eta) \right]$$

Using the optimal controls above, substituting in (A2) and solving the equation, we obtain the optimal control α_θ .

9.2 Proof of Theorem 4

We will start this proof by presenting a definition of viscosity solution for second order partial differential equation in the context of our model. Then, we will prove that v is a viscosity subsolution

and, finally, we will prove that it is also a supersolution of the HJB equation.

Definition 7 *A locally bounded function $v \in USC[\lambda, \infty)$ is a viscosity subsolution of the HJB equation for the transformed problem if and only if, for every $\phi \in C^2([\lambda, \infty))$, and every global maximum point $z_0 \in [\lambda, \infty)$ of the function $v - \phi$*

$$\max \left\{ H(z_0, v, \phi', \phi''), \frac{(z_0 - \lambda)^{1-\gamma}}{1-\gamma} M - v(z_0) \right\} \geq 0$$

A locally bounded function $v \in LSC[\lambda, \infty)$ is a viscosity supersolution of the HJB equation for the transformed problem if and only if, for every $\phi \in C^2([\lambda, \infty))$, and every global minimum point $z_0 \in [\lambda, \infty)$ of the function $v - \phi$

$$\max \left\{ H(z_0, v, \phi', \phi''), \frac{(z_0 - \lambda)^{1-\gamma}}{1-\gamma} M - v(z_0) \right\} \leq 0$$

A continuous function $v : [\lambda, \infty) \rightarrow \mathbb{R}$ is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

9.2.1 v is a viscosity subsolution

We will prove this result by contradiction. Let us assume that $v(z_0) = \phi(z_0)$ and $v(z) \leq \phi(z)$ for every $z \in B(z_0, \psi_1)$, where $B(z_0, \psi_1)$ represents the ball centered at z_0 with radius ψ_1 . If v is not a subsolution then $H(z_0, v, \phi', \phi'') \leq -\varepsilon_1$ and

$$\frac{M}{1-\gamma} (z_0 - \lambda)^{1-\gamma} - v(z_0) \leq -\varepsilon_1$$

for some $\varepsilon_1 > 0$. Given that $\phi \in C^2([\lambda, \infty))$, there is a $B(z_0, \psi_2)$ such that, for all $z \in B(z_0, \psi_2)$ we have $H(z, v, \phi', \phi'') \leq -\varepsilon_1$ and

$$\frac{M}{1-\gamma} (z - \lambda)^{1-\gamma} - v(z_0) \leq -\varepsilon_1$$

Let us define t_1 as the first jumping time of the Lévy process. Then, given that this process is right continuous, either $t_1 = 0$, *a.s.*, or $t_1 > 0$, *a.s.*. For the latter hypothesis define the stopping-time τ_0 as

$$\tau_0 = \inf \{t_1, \inf t : z \notin B(z_0, \psi_2), \inf t : z > R\}$$

for a positive constant $R > z_0$. Using Itô's lemma we get

$$E \left[e^{-\bar{\rho}\tau_0} P_{\tau_0}^{\beta(1-\gamma)} \phi(z_{\tau_0}) \right] = P_0^{\beta(1-\gamma)} \phi(z_0) + E \left[\int_0^{\tau_0} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} (\mathcal{L}\phi(Z_t^*) - \bar{\rho}\phi(Z_t^*)) dt \right],$$

where Z_t^* represents the value of the state variable Z at time t when the optimal controls are used.

Therefore

$$\begin{aligned} E \left[e^{-\bar{\rho}\tau_0} P_{\tau_0}^{\beta(1-\gamma)} \phi(z_{\tau_0}) \right] &\leq P_0^{\beta(1-\gamma)} \phi(z_0) + E \left[\int_0^{\tau_0} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} \left(H(Z_t^*, \phi, \phi', \phi'') - \frac{\widehat{C}_t^{\beta(1-\gamma)}}{1-\gamma} \right) dt \right] \\ &\leq P_0^{\beta(1-\gamma)} \phi(z_0) + E \left[\int_0^{\tau_0} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} \left(-\varepsilon_1 - \frac{\widehat{C}_t^{\beta(1-\gamma)}}{1-\gamma} \right) dt \right] \end{aligned}$$

Defining $\kappa = E \left[\int_0^{\tau_0} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} (\varepsilon_1) dt \right] > 0$, and using the fact that $v(z_0) = \phi(z_0)$ and $v(z_{\tau_0}) \leq \phi(z_{\tau_0})$, we have

$$P_0^{\beta(1-\gamma)} v(z_0) > E \left[e^{-\bar{\rho}\tau_0} P_{\tau_0}^{\beta(1-\gamma)} v(z_{\tau_0}) + \int_0^{\tau_0} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} \left(\frac{\widehat{C}_t^{\beta(1-\gamma)}}{1-\gamma} \right) dt \right] + \kappa$$

which violates the dynamic programming principle.

Now let us consider the case $t_1 = 0$ a.s.. If the jump causes the agent to trade the durable good, then using the dynamic programming principle with $t = 0$ we have

$$P_0^{\beta(1-\gamma)} v(z_0) = P_0^{\beta(1-\gamma)} \frac{M}{1-\gamma} (z_0 - \lambda)^{1-\gamma} \leq P_0^{\beta(1-\gamma)} (z_0 - \lambda)^{1-\gamma} (z^*)^{\gamma-1} \phi(z^*)$$

where $z^* = \arg \max \{ z^{\gamma-1} \phi(z) \}$.

9.2.2 v is a viscosity supersolution

We must prove that

$$0 \geq H(z_0, v, \phi', \phi'') \quad (\text{A3})$$

and

$$0 \geq \frac{M}{1-\gamma} (z_0 - \lambda)^{1-\gamma} - v(z_0) \quad (\text{A4})$$

Inequality (A4) always holds because the agent can always trade the durable good. We must prove that (A3) also holds. Let τ_ζ denote the first exit time from the closed ball $N(\zeta, z_0)$ centered at z_0 , and strictly contained in the solvency region. Let $(\widehat{\Theta}_{1t}, \widehat{C}_t)$ be an admissible policy with $(\widehat{\Theta}_{1t}, \widehat{C}_t) = (\theta_1, c)$ for $t \in [0, \tau_\zeta]$. Using the dynamic programming principle we have

$$P_0^{\beta(1-\gamma)} v(z_0) \geq E \left[\int_0^{h \wedge \tau_\zeta} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} \left(\frac{c^{(1-\gamma)}}{1-\gamma} \right) dt + e^{-\bar{\rho}\tau_\zeta} P_{\tau_\zeta}^{\beta(1-\gamma)} v(z_{\tau_\zeta}) \right] \quad (\text{A5})$$

Using Itô's lemma and taking expectations

$$P_0^{\beta(1-\gamma)}\phi(z_0) = E \left[e^{-\bar{\rho}\tau_\zeta} P_{\tau_\zeta}^{\beta(1-\gamma)}\phi(z_{\tau_\zeta}) - \int_0^{h \wedge \tau_\zeta} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} (\mathcal{L}\phi(Z_t) - \bar{\rho}\phi(Z_t)) dt \right] \quad (A6)$$

Considering that $v(z_0) = \phi(z_0)$ and $v(z_{\tau_0}) \geq \phi(z_{\tau_0})$, and using (A5) and (A6)

$$\begin{aligned} P_0^{\beta(1-\gamma)}\phi(z_0) &\geq E \left[\int_0^{h \wedge \tau_\zeta} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} \left(\frac{c^{\beta(1-\gamma)}}{1-\gamma} \right) dt + e^{-\bar{\rho}\tau_\zeta} P_{\tau_\zeta}^{\beta(1-\gamma)}\phi(z_{\tau_\zeta}) \right] \\ &= P_0^{\beta(1-\gamma)}\phi(z_0) + E \left[\int_0^{h \wedge \tau_\zeta} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} (\mathcal{L}\phi(Z_t) - \bar{\rho}\phi(Z_t)) dt \right. \\ &\quad \left. + \int_0^{h \wedge \tau_\zeta} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} \left(\frac{c^{\beta(1-\gamma)}}{1-\gamma} \right) dt \right] \end{aligned}$$

Therefore

$$0 \geq E \left[\int_0^{h \wedge \tau_\zeta} e^{-\bar{\rho}t} P_t^{\beta(1-\gamma)} \left(\frac{c^{\beta(1-\gamma)}}{1-\gamma} + \mathcal{L}\phi(Z_t) - \bar{\rho}\phi(Z_t) \right) dt \right]$$

Now, letting $h \rightarrow 0$ we get

$$0 \geq \frac{c^{\beta(1-\gamma)}}{1-\gamma} + \mathcal{L}\phi(Z_t) - \bar{\rho}\phi(Z_t)$$

We know that the value function is continuous in (λ, ∞) , because a concave function is continuous on the interior of its domain. Therefore, we are just left to prove that the value function is continuous at the boundary. We will base our proof in the following lemma.

Lemma 8 *For any $\lambda > 0$, there exists constants $\xi > 0$, $0 < \kappa \leq \lambda + \xi$ and $\alpha > 0$, such that*

$$V(x, k, p) \leq \psi(x, k, p) \equiv \frac{\alpha p^{-(1-\beta)(1-\gamma)}}{1-\gamma} (x - \lambda kp)^{1-\gamma}, \quad \text{for every } (x, k, p) \in I_\xi \cup \Gamma_\kappa$$

where

$$I_\xi = \{(x, k, p) \in \mathbb{R}_{++}^3 : \lambda kp \leq x < (\lambda + \xi) kp\} \quad \text{and} \quad \Gamma_\kappa = \{(x, k, p) \in \mathbb{R}_{++}^3 : kp \leq 1, \lambda kp \leq x < \kappa\}$$

Note that this result implies continuity of the value function at the boundary. If $x = \lambda kp$ the only admissible strategy is to liquidate the stock of durable good, which implies $V = 0$. On the other hand, note also that $\psi(x, k, p) \rightarrow 0$, when $(x, k, p) \rightarrow (x_0, k_0, p_0)$ through $I_\xi \cup \Gamma_\kappa$, where (x_0, k_0, p_0) lies in the boundary of the solvency region. Then, we get $0 \leq V(x, k, p) \leq \psi(x, k, p) \rightarrow 0$, for

$$(x, k, p) \rightarrow (x_0, k_0, p_0).$$

Proof. Let $Y_t = (X_t, K_t, P_t)$ and $y = (x, k, p)$ and define

$$\begin{aligned} & F(y, \psi, D\psi, D^2\psi) \\ &= \sup_{(c, \theta)} \left\{ A^{c, \theta} \psi(y) + U(c, k) - \rho \psi(y) \right\} \\ &= \sup_{(c, \theta)} \left\{ \begin{aligned} & \frac{\partial \psi}{\partial x} [r(x - \lambda kp) + \theta(\mu - r) + (\mu_P - \delta)kp - c] + \frac{\partial \psi}{\partial k} \delta k + \frac{\partial \psi}{\partial p} \mu_P p + \frac{1}{2} \frac{\partial^2 \psi}{\partial p^2} (\sigma_{P_1}^2 + \sigma_{P_2}^2) p^2 \\ & + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} [\theta^2 \sigma^2 + k^2 p^2 (\sigma_{P_1}^2 + \sigma_{P_2}^2) + 2\theta \sigma \sigma_{P_1} kp] + \frac{\partial^2 \psi}{\partial x \partial p} p [\sigma_{P_1} (\theta \sigma + kp \sigma_{P_1}) + \sigma_{P_2}^2 k] \\ & + \int_{-1}^L \left[\psi(x + \theta \eta, k, p) - \psi(x, k, p) - \frac{\partial \psi}{\partial \psi} \theta \eta \right] dq(\eta) + \frac{1}{1-\gamma} c^{\beta(1-\gamma)} k^{(1-\beta)(1-\gamma)} - \rho \psi(x, k, p) \end{aligned} \right\} \end{aligned}$$

Let us define the function $\phi(z) = \alpha(z - \lambda)^{1-\gamma} / (1 - \gamma)$ where α is a constant. Note that $\psi(x, k, p) = k^{1-\gamma} p^{\beta(1-\gamma)} \phi(z)$. Then substituting this relation and the optimal c control in the previous equation

$$F = k^{1-\gamma} p^{\beta(1-\gamma)} \sup_{\theta} \left[\alpha (z - \lambda)^{-\gamma-1} G \right]$$

where

$$G = \begin{bmatrix} (z - \lambda)^2 \left[-\delta + \beta \mu_P - \frac{1}{2} (1 - \beta(1 - \gamma)) (\sigma_{P_1}^2 + \sigma_{P_2}^2) - \frac{\rho}{1-\gamma} \right] \\ + (z - \lambda) \left[\begin{aligned} & r(z - 1) + \theta(\mu - r) - (\mu_P - \delta)(z - 1) + \\ & (1 - \beta(1 - \gamma)) [(z - 1)(\sigma_{P_1}^2 + \sigma_{P_2}^2) - \theta \sigma \sigma_{P_1}] \end{aligned} \right] \\ - \gamma \left[\frac{(1-z)^2}{2} (\sigma_{P_1}^2 + \sigma_{P_2}^2) + (1 - z) \theta \sigma \sigma_{P_1} + \frac{\theta^2 \sigma^2}{2} \right] + \\ (z - \lambda)^{\frac{\gamma}{1-\beta(1-\gamma)}} \left[\frac{1}{1-\gamma} - \left(\frac{\alpha}{\beta} \right)^{\frac{1}{1-\beta(1-\gamma)}} \right] + \\ \int_{-1}^{\infty} \left[\frac{\alpha}{1-\gamma} [(z + \theta \eta - \lambda)^{1-\gamma} - (z - \lambda)^{1-\gamma}] - \gamma (z - \lambda)^{-\gamma} \theta \eta \right] dq(\eta) \end{bmatrix}$$

By the mean value theorem

$$\begin{aligned} & \int_{-1}^{\infty} \left[\frac{\alpha}{1-\gamma} [(z + \theta \eta - \lambda)^{1-\gamma} - (z - \lambda)^{1-\gamma}] - \gamma (z - \lambda)^{-\gamma} \theta \eta \right] dq(\eta) \\ &= \int_{-1}^{\infty} \alpha [(z + \theta \eta v - \lambda)^{-\gamma} - (z - \lambda)^{-\gamma}] \theta \eta dq(\eta) \leq 0 \end{aligned}$$

for some $v \in (0, 1)$. Therefore, for any $\alpha > 0$, there exists constants $\xi > 0$, and $\kappa \in (0, \lambda + \xi]$, such that $F(x, \psi, D\psi, D^2\psi) \leq 0$, for every $x \in I_{\xi} \cup \Gamma_{\kappa}$. Define the stopping-time $\tau_0 = \inf \{t \geq 0 : Y_t \notin I_{\xi} \cup \Gamma_{\kappa}\}$. Then, using Itô's lemma

$$E[e^{-\rho \tau} \psi(Y_{\tau_0})] = \psi(y) + E \left[\int_0^{\tau_0} e^{-\rho t} \left(A^{c, \theta} \psi(Y_t) - \rho \psi(Y_t) \right) dt + \sum_{0 \leq t \leq \tau_0} (\psi(Y_t) - \psi(Y_{t-})) \right]$$

where the last term represents the jumps caused by the durable good trading. If the agent trades the durable good, then $X_t = X_{t-} - \lambda P_t K_t$ and $\psi(Y_t) \leq \psi(Y_{t-})$ for every $t \geq 0$. Therefore

$$E[e^{-\rho \tau} \psi(Y_{\tau_0})] \leq \psi(y) - E \left[\int_0^{\tau_0} e^{-\rho t} U(C_t, K_t) \right]$$

If we chose $\alpha \geq \alpha_v ((\lambda + \xi) / \xi)^{1-\gamma}$, then

$$\begin{aligned} \psi(X_{\tau_0}, K_{\tau_0}, P_{\tau_0}) &= \frac{\alpha P_{\tau_0}^{-(1-\beta)(1-\gamma)}}{1-\gamma} (X_{\tau_0} - \lambda K_{\tau_0} P_{\tau_0})^{1-\gamma} \\ &\geq \frac{\alpha_v P_{\tau_0}^{-(1-\beta)(1-\gamma)}}{1-\gamma} X_{\tau_0}^{1-\gamma} \geq V(X_{\tau_0}, K_{\tau_0}, P_{\tau_0}) \end{aligned}$$

where α_v is given by (7) and the last inequality follows from (9). ■

9.3 Proof of Theorem 5

We assume that

$$\sup_{y \in [\lambda, \infty)} \{ \underline{v}(y) - \bar{v}(y) \} > 0 \quad (\text{A7})$$

and aim to find a contradiction. Using the sublinear growth of $\underline{v}(y)$, we can find a $\kappa > 0$, $\omega > 0$, such that $\underline{v}(y) \leq \kappa(1+y)^\omega$ for all $y \in [\lambda, \infty)$. Let $\hat{\omega} \in (\omega, 1)$, then, by (A7), for sufficiently small $\varepsilon > 0$

$$\sup_{y \in [\lambda, \infty)} \{ \underline{v}(y) - \bar{v}(y) - \varepsilon y^{\hat{\omega}} \} > 0 \quad (\text{A8})$$

Using the fact that $\underline{v}(y)$ and $\bar{v}(y)$ are continuous and increasing, that $\underline{v}(\lambda) = \bar{v}(\lambda) = 0$, and that $\lim_{y \rightarrow \infty} \{ \underline{v}(y) - \bar{v}(y) - \varepsilon y^{\hat{\omega}} \}$, then the supremum in (A8) is attained for a $y^* \in (\lambda, \infty)$

$$\sup_{y \in [\lambda, \infty)} \{ \underline{v}(y) - \bar{v}(y) - \varepsilon y^{\hat{\omega}} \} = \underline{v}(y^*) - \bar{v}(y^*) - \varepsilon y^{*\hat{\omega}}$$

Define the functions $\psi, \varphi : [\lambda, \infty) \times [\lambda, \infty) \rightarrow \mathbb{R}$ with $\varphi(y, z) = (\alpha(z-y) - 4\phi)^4 + \varepsilon z^{\hat{\omega}}$ for $\alpha > 1$, $\varepsilon > 0$, $\phi > 0$, and $\psi(y, z) = \underline{v}(z) - \bar{v}(y) - \phi(y, z)$. In order to continue our proof we shall need the following:

Claim 9 *The function $\psi(y, z)$ is bounded on $[\lambda, \infty) \times [\lambda, \infty)$ and attains its supremum in the compact set $[\lambda, \bar{\kappa}] \times [\lambda, \bar{\kappa}]$, where $\bar{\kappa}$ is a constant independent of α , ϕ and ε . The maximum point (y_α, z_α) has the following properties*

1. $\lim_{\alpha \rightarrow \infty} |z_\alpha - y_\alpha| = 0$
2. $\lim_{\alpha \rightarrow \infty} (\alpha(z - y) - 4\phi)^4 = 0$
3. $\lim_{\varepsilon \rightarrow 0, \alpha \rightarrow \infty} \varepsilon z_\alpha^{\hat{\omega}} = 0$.

Claim 10 (Damgaard et al (2003)) *For all $\alpha > 1$, $\phi > 0$, and $\varepsilon > 0$ there exists numbers $Z < 0$ and $Y > 0$, with $Y + Z > 0$ such that*

$$\hat{\rho} \underline{v}(z_a) \leq G\left(z_a, \underline{v}, \frac{\partial \psi}{\partial z}, Z\right) \quad (\text{A9})$$

$$\widehat{\rho v}(y_a) \leq G\left(y_a, \bar{v}, -\frac{\partial \psi}{\partial y}, -Y\right) \quad (\text{A10})$$

and

$$\begin{pmatrix} Z & 0 \\ 0 & y \end{pmatrix} \leq \begin{pmatrix} \frac{\partial^2 \psi}{\partial z^2} & \frac{\partial^2 \psi}{\partial z \partial y} \\ \frac{\partial^2 \psi}{\partial z \partial y} & \frac{\partial^2 \psi}{\partial y^2} \end{pmatrix} = \begin{pmatrix} A + \varepsilon B & -A \\ -A & A \end{pmatrix}$$

with $A = 12\alpha^2 (\alpha (z_a - y_\alpha) - 4\phi)^2 \geq 0$ and $B = \widehat{\omega} (\widehat{\omega} - 1) z_\alpha^{\widehat{\omega}-2} \leq 0$, where $\widehat{\rho} = \rho - (1 - \gamma) (\beta \mu_P - \frac{1}{2}\beta [1 - \beta (1 - \gamma)])$ and

$$G(z, v, a, b) = \sup_{\theta, c \in A(z)} \left\{ a \left[\begin{aligned} & (z-1) (r - \mu_P + \delta + (1 - \beta (1 - \gamma)) (\sigma_{P_1}^2 + \sigma_{P_2}^2)) \\ & - c + \theta (\mu - r - (1 - \beta (1 - \gamma)) \sigma \sigma_{P_1}) \end{aligned} \right] + \right. \\ \left. \frac{1}{2} b \left[(\theta \sigma - (z-1) \sigma_{P_1})^2 + (z-1)^2 \sigma_{P_2}^2 \right] + \int_{-1}^L [v(z + \theta \eta) - v(z) - a \theta \eta] dq(\eta) + \frac{c^{\beta(1-\gamma)}}{1-\gamma} \right\}$$

Claim 11 $\lim_{\varepsilon \rightarrow 0} \lim_{\phi \rightarrow 0} \lim_{\alpha \rightarrow \infty} \left(\frac{\partial \psi}{\partial z} + \frac{\partial \psi}{\partial y} \right) = 0$ and $\lim_{\varepsilon \rightarrow 0} \lim_{\phi \rightarrow 0} \lim_{\alpha \rightarrow \infty} (\widehat{c}^{\beta(1-\gamma)} - \widetilde{c}^{\beta(1-\gamma)}) = 0$ where \widehat{c} (\widetilde{c}) is the consumption control that maximizes $G\left(z_a, \underline{v}, \frac{\partial \psi}{\partial z}, Z\right)$ ($G\left(y_a, \bar{v}, -\frac{\partial \psi}{\partial y}, -Y\right)$).

Claim 12 $\underline{v}(z_\alpha + \widehat{\theta} \eta) - \underline{v}(z_\alpha) - \frac{\partial \psi}{\partial z} \widehat{\theta} \eta - \left(\bar{v}(y_\alpha + \widetilde{\theta} \eta) - \bar{v}(y_\alpha) + \frac{\partial \psi}{\partial y} \widetilde{\theta} \eta \right) \leq 0$ where

$$\widehat{\theta} = \lim_{\varepsilon \rightarrow 0} \lim_{\phi \rightarrow 0} \lim_{\alpha \rightarrow \infty} \arg \max_{\theta} \left[\begin{aligned} & \frac{\partial \psi}{\partial z} \theta (\mu - r - (1 - \beta (1 - \gamma)) \sigma \sigma_{P_1}) + \frac{1}{2} Z (\theta^2 \sigma^2 - 2 (z_\alpha - 1) \theta \sigma \sigma_{P_1}) \\ & + \int_{-1}^L [\underline{v}(z_\alpha + \theta \eta) - \frac{\partial \psi}{\partial z} \theta \eta] dq(\eta) \end{aligned} \right]$$

and

$$\widetilde{\theta} = \lim_{\varepsilon \rightarrow 0} \lim_{\phi \rightarrow 0} \lim_{\alpha \rightarrow \infty} \arg \max_{\theta} \left[\begin{aligned} & -\frac{\partial \psi}{\partial y} \theta (\mu - r - (1 - \beta (1 - \gamma)) \sigma \sigma_{P_1}) - \frac{1}{2} Y (\theta^2 \sigma^2 - 2 (z_\alpha - 1) \theta \sigma \sigma_{P_1}) \\ & + \int_{-1}^L [\bar{v}(z_\alpha + \theta \eta) + \frac{\partial \psi}{\partial y} \theta \eta] dq(\eta) \end{aligned} \right]$$

.

Using equations (A9) and (A10) we have

$$\begin{aligned}
\widehat{\rho}(\underline{v}(z_\alpha) - \bar{v}(y_\alpha)) &\leq \sup_{\theta, c \in A(z_\alpha)} \left\{ \begin{aligned} &\frac{\partial \psi}{\partial z} \left[(z_\alpha - 1) \begin{pmatrix} r - \mu_P + \delta + (1 - \beta(1 - \gamma)) \\ \times (\sigma_{P_1}^2 + \sigma_{P_2}^2) \end{pmatrix} \right] \\ &- c + \theta (\mu - r - (1 - \beta(1 - \gamma)) \sigma \sigma_{P_1}) \\ &+ \frac{1}{2} Z \left[(\theta \sigma - (z_\alpha - 1) \sigma_{P_1}) + (z_\alpha - 1)^2 \sigma_{P_2}^2 \right] \\ &+ \int_{-1}^L \left[\underline{v}(z_\alpha + \theta \eta) - \underline{v}(z_\alpha) - \frac{\partial \psi}{\partial z} \theta \eta \right] dq(\eta) + \frac{c^{\beta(1-\gamma)}}{1-\gamma} \end{aligned} \right\} \\
&- \sup_{\theta, c \in A(y_\alpha)} \left\{ \begin{aligned} &-\frac{\partial \psi}{\partial y} \left[(y_\alpha - 1) \begin{pmatrix} r - \mu_P + \delta + (1 - \beta(1 - \gamma)) \\ \times (\sigma_{P_1}^2 + \sigma_{P_2}^2) \end{pmatrix} \right] \\ &- c + \theta (\mu - r - (1 - \beta(1 - \gamma)) \sigma \sigma_{P_1}) \\ &-\frac{1}{2} Y \left[(\theta \sigma - (y_\alpha - 1) \sigma_{P_1}) + (y_\alpha - 1)^2 \sigma_{P_2}^2 \right] \\ &+ \int_{-1}^L \left[\bar{v}(y_\alpha + \theta \eta) - \bar{v}(y_\alpha) - \frac{\partial \psi}{\partial z} \theta \eta \right] dq(\eta) + \frac{c^{\beta(1-\gamma)}}{1-\gamma} \end{aligned} \right\}
\end{aligned}$$

Applying the claims above we get $\lim_{\varepsilon \rightarrow 0} \lim_{\phi \rightarrow 0} \lim_{\alpha \rightarrow \infty} \widehat{\rho}(\underline{v}(z_\alpha) - \bar{v}(y_\alpha)) \leq 0$, which contradicts the initial assumption.

The proofs of the first three claims follow closely Benth (2002) and Damgaard et al (2003). Regarding the last claim, note that

$$\begin{aligned}
&\underline{v}(z_\alpha + \widehat{\theta} \eta) - \underline{v}(z_\alpha) - \frac{\partial \psi}{\partial z} \widehat{\theta} \eta - \left(\bar{v}(y_\alpha + \widetilde{\theta} \eta) - \bar{v}(y_\alpha) + \frac{\partial \psi}{\partial y} \widetilde{\theta} \eta \right) \\
&= \psi(z_\alpha + \widehat{\theta} \eta, y_\alpha + \widetilde{\theta} \eta) - \psi(z_\alpha, y_\alpha) + \varepsilon(z_\alpha + \widehat{\theta} \eta)^{\widehat{\omega}} - \varepsilon(z_\alpha)^{\widehat{\omega}} - \frac{\partial \psi}{\partial z} \widehat{\theta} \eta - \frac{\partial \psi}{\partial y} \widetilde{\theta} \eta
\end{aligned}$$

and the limit of the last expression as $\varepsilon \rightarrow 0$, $\phi \rightarrow 0$, and $\alpha \rightarrow \infty$, is non-positive due to claims 1 and 3 and to the fact that $\psi(z_\alpha + \widehat{\theta} \eta, y_\alpha + \widetilde{\theta} \eta) \leq \psi(z_\alpha, y_\alpha)$.

10 Appendix B

Our objective is to find a solution to problem (5)-(6). The solution $v(z)$ must satisfy $v(z) \geq \frac{(z-\lambda)^{1-\gamma}}{1-\gamma} M$, for every z belonging to the solvency region, where M is given in equation (11). Let us define $\underline{z} = \inf \left\{ z \in A(z) : v(z) > \frac{(z-\lambda)^{1-\gamma}}{1-\gamma} M \right\}$ and $\bar{z} = \sup \left\{ z \in A(z) : v(z) > \frac{(z-\lambda)^{1-\gamma}}{1-\gamma} M \right\}$, then we impose the following value matching and smooth pasting conditions

$$v(\underline{z}) = \frac{(\underline{z} - \lambda)^{1-\gamma}}{1-\gamma} M \quad (\text{B1})$$

$$v(\bar{z}) = \frac{(\bar{z} - \lambda)^{1-\gamma}}{1-\gamma} M \quad (\text{B2})$$

$$v'_+(\underline{z}) = (\underline{z} - \lambda)^{-\gamma} M \quad (\text{B3})$$

$$v'_-(\bar{z}) = (\bar{z} - \lambda)^{-\gamma} M \quad (\text{B4})$$

where v'_- , and v'_+ denote the left and right derivatives, respectively. A solution to this problem is given by the value function $v(z)$, and the values $(\underline{z}, \bar{z}, M)$, such that (5)-(6) and the conditions above are satisfied.

We solve this problem using and adaptation of the algorithm used in Damgaard et al (2003) and Grossman and Laroque (1990).

Step 1- Guess \underline{z}

Step 2- Guess $M = M_0$, and solve the problem (5)-(6) through value function iteration⁸ on the interval $(\underline{z}, z_{\max})$. Define $\bar{z} = \inf \left\{ z > \underline{z} : v(z) = \frac{(z-\lambda)^{1-\gamma}}{1-\gamma} M \right\}$ and make $v(z) = \frac{(z-\lambda)^{1-\gamma}}{1-\gamma} M$ for $z \in [\bar{z}, z_{\max}]$. Iterate until convergence and calculate the new M using equation (4). If the new $M = M_0$, proceed to step 3. Otherwise, guess a new M_0 and repeat step 2.

Step 3- Verify if the smooth pasting condition (B4) is satisfied. If (B4) holds we have a solution. Otherwise, return to step 1.

⁸See Kushner and Dupuis (1992), for example.

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